BAYER-MACRÌ DECOMPOSITION ON BRIDGELAND MODULI SPACES OVER SURFACES

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ABSTRACT. We find a decomposition formula of the local Bayer-Macrì map for the nef line bundle theory on the Bridgeland moduli space over surface. If there is a global Bayer-Macrì map, such decomposition gives a precise correspondence from Bridgeland walls to Mori walls. As an application, we compute the nef cone of the Hilbert scheme $S^{[n]}$ of n-points over special kinds of fibered surface S of Picard rank two.

CONTENTS

1. INTRODUCTION

Let S be a smooth projective surface over $\mathbb{C}$. Let M be the Gieseker moduli space of semistable sheaves with fixed Chern character ch over S. One point-view of studying the birational geometry of M is to study the classification of line bundles on M, and different cones inside its real Néron-Severi group $N^1(M)$, such as nef cone $\mathrm{Nef}(M)$, pseudo-effective cone $\overline{\mathrm{Eff}}(M)$. Another point-view is introduced by Bridgeland [Bri07] by the idea of enlarging the category of coherent sheaves $\mathrm{Coh}(S)$ to its bounded derived version $\mathrm{D}^{\mathrm{b}}(S)$, and studying moduli spaces of Bridgeland semistable objects in $\mathrm{D}^{\mathrm{b}}(S)$. Let σ be a Bridgeland stability condition and $M_\sigma(\mathrm{ch})$ be the moduli space of σ-semistable objects in $\mathrm{D}^{\mathrm{b}}(S)$ with the same invariant ch. The collection of all stability conditions forms an interesting parameter space $\mathrm{Stab}(S)$, which is a $\mathbb{C}$-manifold. Bridgeland identified M as $M_\sigma(\mathrm{ch})$, for σ

in a special chamber inside $\mathrm{Stab}(S)$. He envisioned that the wall-chamber structures of $\mathrm{Stab}(S)$ will recover birational models of M.

When S is the projective plane $\mathbb{P}^2$ and $\mathrm{ch} = (1, 0, -n)$, the moduli space M is the Hilbert scheme $\mathbb{P}^{2[n]}$ of n-points over $\mathbb{P}^2$. Arcara, Bertram, Coskun and Huizenga [ABCH13] found a precise relation between Bridgeland walls inside $\mathrm{Stab}(\mathbb{P}^2)$ with respect to the $(1, 0, -n)$ and Mori walls inside the $\overline{\mathrm{Eff}}(\mathbb{P}^{2[n]})$, see Corollary 4.14. Bertram and Coskun generalized the speculation to other rational surfaces [BC13]. Bayer and Macrì [BM14a, BM14b] linked the two point-views by establishing a line bundle theory on Bridgeland moduli spaces. Let $\sigma = \sigma_{\omega,\beta}$ be the stability condition constructed by Arcara and Bertram [AB13], which depends on an ample line bundle ω and another line bundle β over S. Assume that σ is in a chamber C. Bayer and Macrì constructed a map by sending σ to a nef line bundle ℓ_σ on $M_\sigma(\mathrm{ch})$, which is called the local Bayer-Macrì map. The line bundle ℓ_σ only depends on the chamber C. If S is a K3 surface and ch is primitive, they then constructed a global Bayer-Macrì map (by gluing the local Bayer-Macrì maps, see Definition 3.6),

$$\ell : \mathrm{Stab}^\dagger(S) \to N^1(M),$$

sending a stability condition σ to a line bundle ℓ_σ on M. The existence of the global Bayer-Macrì map is also known for the projective plane $\mathbb{P}^2$ with primitive Chern character ch by Li and Zhao [LZ16].

In this paper, we find a decomposition of the line bundle $\ell_{\sigma_{\omega,\beta}}$. The decomposition is classified into two cases according to the given Chern character ch. The case for objects supported in dimension one is given in Lemma 4.5. The case for objects supported in dimension two is given in Lemma 4.8. In this case, an equivalent decomposition is also obtained by Bolognese, Huizenga, Lin, Riedl, Schmidt, Woolf and Zhao [BHL+16, Proposition 3.8.]. If there is a global Bayer-Macrì map, we then obtain the precise correspondences from Bridgeland walls to Mori walls for such two cases, see Theorem 4.10 and Theorem 4.13 respectively. By Mori walls, we mean the walls appear on the stable base locus decomposition of the pseudo-effective cone $\overline{\mathrm{Eff}}(M)$.

As an application of the main theorem, we compute the nef cone of the Hilbert scheme $S^{[n]}$ of n-points over special kinds of fibered surface S in Theorem 5.2. Here S is either the Hirzebruch surface or an elliptic surface over $\mathbb{P}^1$ with a global section of Picard rank two. The example suggests that to obtain extremal nef line bundle, we cannot assume that the ω is parallel to β.

Some of the techniques discussed in this paper have been partially generalized by Coskun and Huizenga [CH15] to compute the nef cone of certain Gieseker moduli spaces.

Outline of the paper. Section 2 is a brief review of the notion of Bridgeland stability condition. Section 3 is a brief review of Bayer and Macrì's line bundle theory on Bridgeland moduli spaces. Main theorems on the Bayer-Macrì decomposition are given in Section 4. In section 5, we provide an application of the main theorem. Some backgrounds on the large volume limit are given as Appendix A. Some parallel computations by using $\hat{Z}_{\omega,\beta}$ (B.1) for K3 surface are given in Appendix B.

2. Bridgeland stability conditions

Let S be a smooth projective surface over $\mathbb{C}$ and $\mathrm{D}^{\mathrm{b}}(S)$ be the bounded derived category of coherent sheaves on S. Denote the Grothendieck group of $\mathrm{D}^{\mathrm{b}}(S)$ by $K(S)$. A *Bridgeland stability condition* ([Bri07, Proposition 5.3]) $\sigma = (Z, \mathcal{A})$ on $\mathrm{D}^{\mathrm{b}}(S)$ consists of a pair $(Z, \mathcal{A})$, where $Z : K(S) \to \mathbb{C}$ is a group homomorphism (called *central charge*) and $\mathcal{A} \subset \mathrm{D}^{\mathrm{b}}(S)$ is the heart of a *bounded t-structure*, satisfying the following three properties.

(1) *Positivity.* For any $0 \neq E \in \mathcal{A}$ the central charge $Z(E)$ lies in the semi-closed upper half-plane $\mathbb{R}_{>0} \cdot \mathrm{e}^{(0,1] \cdot i\pi}$.

Let $E \in \mathcal{A} \setminus \{0\}$. Define the *Bridgeland slope* (might be $+\infty$ valued) and the *phase* of E as

$$\mu_\sigma(E) := \frac{-\Re(Z(E))}{\Im(Z(E))}; \quad \phi(E) := \frac{1}{\pi}\arg(Z(E)) \in (0, 1].$$

For nonzero E, $F \in \mathcal{A}$, we have the equivalent relation:

$$\mu_\sigma(F) < (\leq)\mu_\sigma(E) \Longleftrightarrow \phi(F) < (\leq)\phi(E).$$

For $0 \neq E \in \mathcal{A}$, we say E is *Bridgeland (semi)stable* if for any subobject $0 \neq F \subsetneq E$ ($0 \neq F \subseteq E$) we have $\mu_\sigma(F) < (\leq)\mu_\sigma(E)$.

(2) *Harder-Narasimhan property.* Every object $E \in \mathcal{A}$ has a Harder-Narasimhan filtration $0 = E_0 \hookrightarrow E_1 \hookrightarrow \ldots \hookrightarrow E_n = E$ such that the quotients E_i/E_{i-1} are Bridgeland-semistable, with $\mu_\sigma(E_1/E_0) > \mu_\sigma(E_2/E_1) > \cdots > \mu_\sigma(E_n/E_{n-1})$.

(3) *Support property.* There is a constant $C > 0$ such that, for all Bridgeland-semistable object $E \in \mathcal{A}$, we have $\|E\| \leq C|Z(E)|$, where $\|\cdot\|$ is a fixed norm on $K(X) \otimes \mathbb{R}$.

2.1. **Bridgeland Stability Conditions on Surfaces.** Let S be a smooth projective surface. Fix $\omega, \beta \in N^1(S) := \mathrm{NS}(S)_\mathbb{R}$ with ω ample. Define

$$Z_{\omega,\beta}(E) := -\int_S \mathrm{e}^{-(\beta+\sqrt{-1}\omega)}.\mathrm{ch}(E).$$

For $E \in \mathrm{Coh}(S)$, denote its *Mumford slope* by

$$\mu_\omega(E) := \begin{cases} \frac{\omega \cdot \mathrm{ch}_1(E)}{\mathrm{ch}_0(E)} & \text{if } \mathrm{ch}_0(E) \neq 0; \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\mathcal{T}_{\omega,\beta} \subset \mathrm{Coh}(S)$ be the subcategory of coherent sheaves whose HN-factors (with respect to Mumford stability) are of Mumford slope strictly greater than $\omega.\beta$. Let $\mathcal{F}_{\omega,\beta} \subset \mathrm{Coh}(S)$ be the subcategory of coherent sheaves whose HN-factors (with respect to Mumford stability) are of Mumford slope less than or equal to $\omega.\beta$. Then $(\mathcal{T}_{\omega,\beta}, \mathcal{F}_{\omega,\beta})$ is a torsion pair of $\mathrm{Coh}(S)$ [AB13]. Define the heart $\mathcal{A}_{\omega,\beta}$ as the tilt of such torsion pair:

$$\mathcal{A}_{\omega,\beta} := \{E \in \mathrm{D}^b(S) : H^{-1}(E) \in \mathcal{F}_{\omega,\beta}, H^0(E) \in \mathcal{T}_{\omega,\beta}, H^p(E) = 0 \text{ otherwise}\}.$$

Lemma 2.1. [AB13, Corollary 2.1] *Fix* $\omega, \beta \in \mathrm{NS}(S)_{\mathbb{R}}$ *with* ω *ample. Then* $\sigma_{\omega,\beta} := (Z_{\omega,\beta}, \mathcal{A}_{\omega,\beta})$ *is a Bridgeland stability condition.*

2.2. **Logarithm Todd class.** Let X be a smooth projective variety over $\mathbb{C}$. Let us introduce a formal variable t, and write

$$\begin{aligned}
\mathrm{td}(X)(t) \quad := \quad & 1 + (-\frac{1}{2}\mathrm{K}_X)t + \frac{1}{12}\left(\frac{3}{2}\mathrm{K}_X^2 - \mathrm{ch}_2(X)\right)t^2 \\
& + \left(-\frac{1}{24}\mathrm{K}_X.\left(\frac{1}{2}\mathrm{K}_X^2 - \mathrm{ch}_2(X)\right)\right)t^3 + \text{ higher order of } t^4.
\end{aligned}$$

Taking the logarithm with respect to t, and expressing it in the power series of t, we obtain

$$\ln \mathrm{td}(X)(t) = -\frac{1}{2}\mathrm{K}_X t - \frac{1}{12}\mathrm{ch}_2(X)t^2 + 0 \cdot t^3 + \text{ higher order of } t^4.$$

In particular, the *logarithm Todd class* of a smooth projective surface S or a smooth projective threefold X is given respectively by

$$\begin{aligned}
\ln \mathrm{td}(S) \quad &:= \quad (0, -\frac{1}{2}\mathrm{K}_S, -\frac{1}{12}\mathrm{ch}_2(S)); \text{ or} \\
\ln \mathrm{td}(X) \quad &:= \quad (0, -\frac{1}{2}\mathrm{K}_X, -\frac{1}{12}\mathrm{ch}_2(X), 0).
\end{aligned}$$

2.3. **The Mukai pairing.** We refer to [Huy06, Section 5.2] for the details. Let X still be a smooth projective variety of dimension n over $\mathbb{C}$. Define the Mukai vector of an object $E \in \mathrm{D}^b(X)$ by

$$v(E) := \mathrm{ch}(E).\mathrm{e}^{\frac{1}{2}\ln \mathrm{td}(X)} \in \oplus H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q}) =: H_{\mathrm{alg}}^*(X, \mathbb{Q}).$$

Let $A(X)$ be the Chow ring of X. The Chern character gives a mapping $\mathrm{ch} : K(X) \to A(X) \otimes \mathbb{Q}$. There is a natural involution $* : A(X) \to A(X)$,

$$v = (v_0, \ldots, v_i, \ldots, v_n) \mapsto v^* := (v_0, \ldots, (-1)^i v_i, \ldots, (-1)^n v_n).$$

We call v^* the *Mukai dual* of v. Denote $E^{\vee} := R\mathcal{H}om(E, \mathcal{O}_S)$. We have

$$\mathrm{ch}(E^{\vee}) = (\mathrm{ch}(E))^*, \quad v(E^{\vee}) = (v(E))^*.\mathrm{e}^{-\frac{1}{2}\mathrm{K}_X}.$$

Define the *Mukai pairing* for two Mukai vectors w and v by

$$(2.1) \qquad\qquad \langle w, v \rangle_X := -\int_X w^*.v.\mathrm{e}^{-\frac{1}{2}\mathrm{K}_X}.$$

The Hirzebruch-Riemann-Roch theorem gives

$$\chi(F, E) = \int_X \mathrm{ch}(F^{\vee}).\mathrm{ch}(E).\mathrm{td}(X) = -\langle v(F), v(E) \rangle_X.$$

For a smooth projective surface S, the Mukai vector of $E \in \mathrm{D}^b(S)$ is

$$
\begin{aligned}
v(E) &= (v_0(E), v_1(E), v_2(E)) \\
(2.2) \quad &= \left(\mathrm{ch}_0, \mathrm{ch}_1 - \frac{1}{4}\mathrm{ch}_0 \mathrm{K}_S, \mathrm{ch}_2 - \frac{1}{4}\mathrm{ch}_1.\mathrm{K}_S + \frac{1}{2}\mathrm{ch}_0\left(\chi(\mathcal{O}_S) - \frac{1}{16}\mathrm{K}_S^2\right)\right).
\end{aligned}
$$

By (2.1) the Mukai paring of $w = (w_0, w_1, w_2)$ and $v = (v_0, v_1, v_2)$ is

$$
(2.3) \quad \langle w,\ v \rangle_S = w_1.v_1 - w_0\left(v_2 - \frac{1}{2}v_1.\mathrm{K}_S\right) - v_0\left(w_2 + \frac{1}{2}w_1.\mathrm{K}_S\right) - \frac{1}{8}w_0 v_0 \mathrm{K}_S^2.
$$

2.4. Central charge in terms of the Mukai paring.

Lemma 2.2. *The central charge $Z_{\omega,\beta}$ has the expression:*

$$
(2.4) \quad Z_{\omega,\beta}(E) = \langle \mho_{Z_{\omega,\beta}}, v(E) \rangle_S, \quad \text{where } \mho_{Z_{\omega,\beta}} := e^{\beta - \frac{3}{4}\mathrm{K}_S + \sqrt{-1}\omega + \frac{1}{24}\mathrm{ch}_2(S)}.
$$

Moreover the vector $\mho_{Z_{\omega,\beta}}$ (or simply $\mho_Z$) is given by

$$
\begin{aligned}
\mho_{Z_{\omega,\beta}} &= \left(1, \beta - \frac{3}{4}\mathrm{K}_S, -\frac{1}{2}\omega^2 + \frac{1}{2}(\beta - \frac{3}{4}\mathrm{K}_S)^2 - \frac{1}{2}\left(\chi(\mathcal{O}_S) - \frac{1}{8}\mathrm{K}_S^2\right)\right) \\
(2.5) \quad &+ \sqrt{-1}\left(0, \omega, (\beta - \frac{3}{4}\mathrm{K}_S).\omega\right).
\end{aligned}
$$

Proof.

$$
\begin{aligned}
Z_{\omega,\beta}(E) &= -\int_S e^{-(\beta + \sqrt{-1}\omega)}.\mathrm{ch}(E) \\
&= -\int_S e^{-(\beta + \ln \mathrm{td}(S) + \sqrt{-1}\omega)}.\sqrt{\mathrm{td}(S)}.\mathrm{ch}(E).\sqrt{\mathrm{td}(S)}.
\end{aligned}
$$

Denote $\mathrm{ch}(F^\vee) := e^{-(\beta + \ln \mathrm{td}(S) + \sqrt{-1}\omega)}$. Then

$$
\begin{aligned}
\mathrm{ch}(F)^* &= \mathrm{ch}(F^\vee) = e^{-(\beta - \frac{1}{2}\mathrm{K}_S + \sqrt{-1}\omega) + \frac{1}{12}\mathrm{ch}_2(S)} \\
&= \left(1, -(\beta - \frac{1}{2}\mathrm{K}_S + \sqrt{-1}\omega), \frac{1}{2}(\beta - \frac{1}{2}\mathrm{K}_S + \sqrt{-1}\omega)^2 + \frac{1}{12}\mathrm{ch}_2(S)\right).
\end{aligned}
$$

So

$$
\begin{aligned}
\mathrm{ch}(F) &= \left(1, (\beta - \frac{1}{2}\mathrm{K}_S + \sqrt{-1}\omega), \frac{1}{2}(\beta - \frac{1}{2}\mathrm{K}_S + \sqrt{-1}\omega)^2 + \frac{1}{12}\mathrm{ch}_2(S)\right) \\
&= e^{(\beta - \frac{1}{2}\mathrm{K}_S + \sqrt{-1}\omega) + \frac{1}{12}\mathrm{ch}_2(S)}.
\end{aligned}
$$

Therefore,

$$
Z_{\omega,\beta}(E) = -\int_S \mathrm{ch}(F^\vee).\sqrt{\mathrm{td}(S)}.\mathrm{ch}(E).\sqrt{\mathrm{td}(S)} = \langle v(F), v(E) \rangle_S.
$$

So

$$
\mho_{Z_{\omega,\beta}} = v(F) = \mathrm{ch}(F).e^{\frac{1}{2}\ln \mathrm{td}(S)} = e^{\beta - \frac{3}{4}\mathrm{K}_S + \sqrt{-1}\omega + \frac{1}{24}\mathrm{ch}_2(S)}.
$$

By using the Noether's formula

$$
-\frac{1}{12}\mathrm{ch}_2(S) = \chi(\mathcal{O}_S) - \frac{1}{8}\mathrm{K}_S^2.
$$

and direct computation, we get the concrete expression of $\mho_Z$. $\qquad\square$

Denote $\mathrm{Stab}(S)$ the collection of all Bridgeland stability conditions. It is a $\mathbb{C}$-manifold of dimension $K_{\mathrm{num}}(S) \otimes \mathbb{C}$, with two group actions: a left action by $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(S))$ and a right action by $\widetilde{\mathrm{GL}_2^+}(\mathbb{R})$ [Bri07, Lemma 8.2]. The stability σ is said to be *geometric* if all skyscraper sheaves $\mathcal{O}_x$, $x \in S$, are σ-stable of the same phase. We can set the phase to be 1 by a right group action. Denote by $\mathrm{Stab}^\dagger(S) \subset \mathrm{Stab}(S)$ the connected component containing geometric stability conditions. The stability σ is said to be *numerical* if the central charge Z takes the form $Z(E) = \langle \pi(\sigma), v(E) \rangle_S$ for some vector $\pi(\sigma) \in K_{\mathrm{num}}(S) \otimes \mathbb{C}$. As in [Huy14, Remark 4.33], we further *assume* the numerical Bridgeland stability factors through $K_{\mathrm{num}}(S)_{\mathbb{Q}} \otimes \mathbb{C} \to H_{\mathrm{alg}}^*(S, \mathbb{Q}) \otimes \mathbb{C}$. Therefore $\pi(\sigma) \in H_{\mathrm{alg}}^*(S, \mathbb{Q}) \otimes \mathbb{C}$. For a numerical geometric stability condition with skyscraper sheaves of phase 1, the heart $\mathcal{A}$ must be of the form $\mathcal{A}_{\omega,\beta}$ (see [Bri07, Proposition 10.3] and Huybrechts [Huy14, Theorem 4.39]). Therefore, Lemma 2.2 gives

$$(2.6) \qquad \pi(\sigma_{\omega,\beta}) = \mho_{Z_{\omega,\beta}} \in H_{\mathrm{alg}}^*(S, \mathbb{Q}) \otimes \mathbb{C}.$$

2.5. **Bertram's nested wall theorem.** We follow notations in [Mac14, Section 2] (but use H instead of ω therein). Fix an ample divisor H and another divisor $\gamma \in H^\perp$, i.e. $H.\gamma = 0$. Denote

$$g := H^2, \quad -d := \gamma^2.$$

It is known by *Hodge index theorem* that $d \geq 0$, and $d = 0$ if and only if $\gamma = 0$. Let $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$ be of Bogomolov type, i.e. $\mathrm{ch}_1^2 - 2\mathrm{ch}_0 \mathrm{ch}_2 \geq 0$. Write it as

$$\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2) := (x, y_1 H + y_2 \gamma + \delta, z),$$

where y_1, y_2 are real coefficients, and $\delta \in \{H, \gamma\}^\perp$. Write the potential destabilizing Chern character as

$$\mathrm{ch}' = (\mathrm{ch}_0', \mathrm{ch}_1', \mathrm{ch}_2') := (r, c_1 H + c_2 \gamma + \delta', \chi),$$

where c_1, c_2 are real coefficients, and $\delta' \in \{H, \gamma\}^\perp$. A *potential wall* is defined as

$$W(\mathrm{ch}, \mathrm{ch}') := \{\sigma \in \mathrm{Stab}(S) \mid \mu_\sigma(\mathrm{ch}) = \mu_\sigma(\mathrm{ch}')\}.$$

A potential wall $W(\mathrm{ch}, \mathrm{ch}')$ is a *Bridgeland wall* if there is a $\sigma \in \mathrm{Stab}(S)$ and objects E, $F \in \mathcal{A}_\sigma$ such that $\mathrm{ch}(E) = \mathrm{ch}$, $\mathrm{ch}(F) = \mathrm{ch}'$ and $\mu_\sigma(E) = \mu_\sigma(F)$. There is a wall-chamber structure on $\mathrm{Stab}(S)$ with respect to ch [Bri07, Bri08, Tod08]. Bridgeland walls are real codimension 1 in $\mathrm{Stab}(S)$, which separate $\mathrm{Stab}(S)$ into chambers. Let E be an object that is σ_0-stable for a stability condition σ_0 in some chamber $\mathtt{C}$. Then E is σ-stable for any $\sigma \in \mathtt{C}$. Choose

$$(2.7) \qquad \begin{cases} \omega := tH, \\ \beta := sH + u\gamma, \end{cases}$$

for some real numbers t, s, u, with t positive. With a sign choice of γ, we further assume $u \geq 0$. There is a half 3-space of stability conditions

$$\Omega_{\omega,\beta} = \Omega_{tH, sH+u\gamma} := \{\sigma_{tH, sH+u\gamma} \mid t > 0, u \geq 0\} \subset \mathrm{Stab}^\dagger(S),$$

which should be considered as the u-indexed family of half planes

$$\Pi_{(H, \gamma, u)} := \{\sigma_{tH, sH+u\gamma} \mid t > 0, \ u \text{ is fixed.}\}.$$

Definition 2.3. A *frame* with respect to the triple (H, γ, u), is a choice of an ample divisor H on S, another divisor $\gamma \in H^{\perp}$, and non-negative number u, such that the stability conditions $\sigma_{\omega, \beta}$ are on the half plane $\Pi_{(H, \gamma, u)}$ with (s, t)-coordinates as equation (2.7). We simply call this as fixing a frame (H, γ, u), and write $\sigma_{s,t} := \sigma_{tH, sH+u\gamma}$.

Theorem 2.4 (Bertram's nested wall theorem in (s, t)-model). [Mac14, Section 2] *Fix a frame (H, γ, u). The potential walls $W(\mathrm{ch}, \mathrm{ch}')$ (for the fixed ch and different potential destabilizing Chern character ch') in the (s,t)-half-plane $\Pi_{(H, \gamma, u)}$ $(t > 0)$ are given by nested semicircles with center $(C, 0)$ and radius $R = \sqrt{D + C^2}$:*

$$(2.8) \qquad (s - C)^2 + t^2 = D + C^2,$$

where $C = C(\mathrm{ch}, \mathrm{ch}')$ and $D = D(\mathrm{ch}, \mathrm{ch}')$ are given by

$$(2.9) \qquad C(\mathrm{ch}, \mathrm{ch}') := \frac{x\chi - rz + ud(xc_2 - ry_2)}{g(xc_1 - ry_1)},$$

$$(2.10) \quad D(\mathrm{ch}, \mathrm{ch}') := \frac{2zc_1 - 2c_2 u d y_1 - x u^2 d c_1 + 2 y_2 u d c_1 - 2\chi y_1 + r u^2 d y_1}{g(xc_1 - ry_1)}.$$

- *If $\mathrm{ch}_0 = x \neq 0$, we have*

$$(2.11) \qquad D = -\frac{2y_1}{x}C + \frac{ud(2y_2 - ux) + 2z}{gx}$$

$$(2.12) \qquad = -\frac{2y_1}{x}C + \left(\frac{y_1^2}{x^2} - F\right),$$

   where $F = F(\mathrm{ch})$ is independent of ch':

$$(2.13) \qquad F(\mathrm{ch}) := \frac{d}{g}\left(u - \frac{y_2}{x}\right)^2 + \frac{1}{x^2 g}(y_1^2 g - y_2^2 d - 2xz).$$

   Moreover, if ch is of Bogomolov type, i.e. $\mathrm{ch}_1^2 - 2\mathrm{ch}_0\mathrm{ch}_2 \geq 0$, then $F(\mathrm{ch}) \geq 0$ for all u.
- *If $\mathrm{ch}_0 = 0$ and $\mathrm{ch}_1.H > 0$, i.e. $x = 0$ and $y_1 > 0$, then $\mathrm{ch}_0' = r \neq 0$, and $C = \frac{z + duy_2}{gy_1}$ is independent of ch'. We have*

$$(2.14) \qquad D = -\frac{2c_1}{r}C + \frac{ud(2c_2 - ur) + 2\chi}{gr}$$

$$(2.15) \qquad = -\frac{2c_1}{r}C + \left(\frac{c_1^2}{r^2} - F'\right),$$

   where $F' = F'(\mathrm{ch}')$ is independent of ch:

$$(2.16) \qquad F'(\mathrm{ch}') := \frac{d}{g}\left(u - \frac{c_2}{r}\right)^2 + \frac{1}{r^2 g}(c_1^2 g - c_2^2 d - 2r\chi).$$

   Moreover, if ch' is of Bogomolov type, then $F(\mathrm{ch}') \geq 0$ for all u.

Proof. We refer to Maciocia's paper [Mac14, Section 2]. The only unproved parts are equations (2.14, 2.15). It is an easy exercise to check them. $\square$

2.6. **From (s,t)-model to (s,q)-model.** We follow the idea of Li-Zhao [LZ16], and consider a $\widetilde{\mathrm{GL}_2^+}(\mathbb{R})$ action on $\sigma_{\omega,\beta}$. The potential walls in the (s,q)-plane are semi-lines.

Definition 2.5. Define $\sigma'_{\omega,\beta} = (Z'_{\omega,\beta}, \mathcal{A}'_{\omega,\beta})$ as the right action of $\begin{pmatrix} 1 & 0 \\ -\frac{s}{t} & \frac{1}{t} \end{pmatrix}$ on $\sigma_{\omega,\beta}$, i.e. $\mathcal{A}'_{\omega,\beta} = \mathcal{A}_{\omega,\beta}$ and

$$(2.17) \qquad Z'_{\omega,\beta}(E) := \left(\Re Z_{\omega,\beta}(E) - \frac{s}{t} \Im Z_{\omega,\beta}(E) \right) + \frac{1}{t} i \Im Z_{\omega,\beta}(E).$$

Lemma 2.6. *Fix a frame (H, γ, u). The above right action does not change the potential walls $W(\mathrm{ch}, \mathrm{ch}')$ in the (s,t)-plane $\Pi_{(H,\gamma,u)}$.*

Proof. This is a direct computation because the potential wall relation for $Z'_{\omega,\beta}$ is equivalent to the potential wall relation for $Z_{\omega,\beta}$ by using (2.17):

$$\Re Z'(\mathrm{ch}') \Im Z'(\mathrm{ch}) - \Re Z'(\mathrm{ch}) \Im Z'(\mathrm{ch}') = 0$$
$$\Leftrightarrow \quad \Re Z(\mathrm{ch}') \Im Z(\mathrm{ch}) - \Re Z(\mathrm{ch}) \Im Z(\mathrm{ch}') = 0.$$

$\square$

Definition 2.7. Fix a frame (H, γ, u). We change the (s,t)-plane $\Pi_{(H,\gamma,u)}$ to the (s,q)-plane $\Sigma_{(H,\gamma,u)}$ by keeping the same s and defining

$$(2.18) \qquad\qquad\qquad q := \frac{s^2 + t^2}{2}.$$

Denote $\sigma_{s,q} := \sigma'_{tH, sH+u\gamma}$. The central charge (2.17) becomes

$$
\begin{aligned}
Z_{s,q}(E) &= (-\mathrm{ch}_2(E) + \mathrm{ch}_0(E)H^2 q) + \left(-\frac{1}{2}\mathrm{ch}_0(E)\gamma^2 u^2 + u\mathrm{ch}_1(E).\gamma \right) \\
&\quad + i(\mathrm{ch}_1(E).H - \mathrm{ch}_0(E)H^2 s).
\end{aligned}
$$

Corollary 2.8 (Bertram's nested wall theorem in (s,q)-model)**.** *Fix a frame (H, γ, u) and use notations as above. The potential walls $W(\mathrm{ch}, \mathrm{ch}')$ in the (s,q)-plane $\Sigma_{(H,\gamma,u)}$ are given by semi-lines*

$$q = Cs + \frac{1}{2}D, \quad (q > \frac{s^2}{2}).$$

- *If $x \neq 0$, then the potential walls are given by semi-lines passing through a fixed point $(\frac{y_1}{x}, \frac{1}{2}\left(\frac{y_1^2}{x^2} - F \right))$ with slope $C = C(\mathrm{ch}, \mathrm{ch}')$:*

$$(2.19) \qquad q = C(s - \frac{y_1}{x}) + \frac{1}{2}\left(\frac{y_1^2}{x^2} - F \right), \quad (q > \frac{s^2}{2}),$$

  where $F = F(\mathrm{ch})$ as in equation (2.13) is independent of ch'.
- *If $x = 0$ and $y_1 > 0$, then $r \neq 0$. The potential walls are given by parallel semi-lines with constant slope $C = \frac{z + duy_2}{gy_1}$:*

$$(2.20) \qquad q = C(s - \frac{c_1}{r}) + \frac{1}{2}\left(\frac{c_1^2}{r^2} - F' \right), \quad (q > \frac{s^2}{2}),$$

  where $F' = F'(\mathrm{ch}')$ as in equation (2.16) is independent of ch.

Proof. This is a direct computation by using equations (2.8, 2.18). $\square$

Remark 2.9. In the case of $\mathbb{P}^2$, the condition $q > \frac{s^2}{2}$ is relaxed, and q could be a little negative and the boundary is given by a fractal curve [LZ16].

2.7. **Duality induced by derived dual.**

Lemma 2.10. [Mar13, Theorem 3.1] *The functor* $\Phi(\cdot) := R\mathcal{H}om(\cdot, \mathcal{O}_S)[1]$ *induces an isomorphism between the Bridgeland moduli spaces* $M_{\omega,\beta}(\mathrm{ch})$ *and* $M_{\omega,-\beta}(-\mathrm{ch}^*)$ *provided these moduli spaces exist and* $Z_{\omega,\beta}(\mathrm{ch})$ *belongs to the open upper half plane.*

Proof. This is a variation of Martinez's duality theorem [Mar13, Theorem 3.1], where the duality functor is taken as $R\mathcal{H}om(\cdot, \omega_S)[1]$. $\square$

Corollary 2.11. *Fix the Chern character* $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$. *Assume that* $Z_{\omega,\beta}(\mathrm{ch})$ *belongs to the open upper half plane. The wall-chamber structures of* $\sigma_{\omega,\beta}$ *w.r.t.* ch *is dual to the wall-chamber structures of* $\Phi(\sigma_{\omega,\beta})$ *w.r.t.* $\Phi(\mathrm{ch}) = -\mathrm{ch}^* = (-\mathrm{ch}_0, \mathrm{ch}_1, -\mathrm{ch}_2)$ *in the sense that*

$$\Phi(\sigma_{\omega,\beta}) = \sigma_{\omega,-\beta}.$$

Applying Φ *again, we have* $\Phi \circ \Phi(\sigma_{\omega,\beta}) = \sigma_{\omega,\beta}$. *Moreover, if we fix a frame* (H, γ, u), *then* $\sigma_{\omega,\beta} \in \Pi_{(H,\gamma,u)}$ *with coordinates* (s,t) *is dual to* $\Phi(\sigma_{\omega,\beta}) \in \Pi_{(H,-\gamma,u)}$ *with coordinates* $(-s,t)$.

- *If* $\sigma_{\omega,\beta} \in \mathtt{C}$, *where* $\mathtt{C}$ *is a chamber w.r.t.* ch *in* $\Pi_{(H,\gamma,u)}$, *then we have* $\Phi(\sigma_{\omega,\beta}) \in \mathtt{DC}$, *where* $\mathtt{DC}$ *is the corresponding chamber w.r.t.* $\Phi(\mathrm{ch})$ *in* $\Pi_{(H,-\gamma,u)}$.
- *If* $\sigma := \sigma_{\omega,\beta} \in W(\mathrm{ch}, \mathrm{ch}')$ *in* $\Pi_{(H,\gamma,u)}$, *then* $\Phi(\sigma) \in W(-\mathrm{ch}^*, -\mathrm{ch}'^*)$ *in* $\Pi_{(H,-\gamma,u)}$, *and there are relations:*
  $\mu_{\Phi(\sigma)}(-\mathrm{ch}^*) = -\mu_\sigma(\mathrm{ch})$; $C_{\Phi(\sigma)}(-\mathrm{ch}^*, -\mathrm{ch}'^*) = -C_\sigma(\mathrm{ch}, \mathrm{ch}')$;
  $D_{\Phi(\sigma)}(-\mathrm{ch}^*, -\mathrm{ch}'^*) = D_\sigma(\mathrm{ch}, \mathrm{ch}')$; $R_{\Phi(\sigma)}(-\mathrm{ch}^*, -\mathrm{ch}'^*) = R_\sigma(\mathrm{ch}, \mathrm{ch}')$.

Proof. The proof is a direct computation. $\square$

Remark 2.12. The assumption that $Z_{\omega,\beta}(\mathrm{ch})$ belongs to the open upper half plane means exactly that we exclude the case $\Im Z_{\omega,\beta}(\mathrm{ch}) = 0$, which is equivalent to one of the following three subcases:

- $\mathrm{ch} = (0, 0, n)$ for some positive integer n; or
- $\mathrm{ch}_0 > 0$ and $\Im Z_{\omega,\beta}(\mathrm{ch}) = 0$; or
- $\mathrm{ch}_0 < 0$ and $\Im Z_{\omega,\beta}(\mathrm{ch}) = 0$.

We call the first subcase as the trivial chamber, the second subcase as the Uhlenbeck wall and the third subcase as the dual Uhlenbeck wall, see Definition A.2.

3. BAYER-MACRÌ'S NEF LINE BUNDLE THEORY

3.1. **The local Bayer-Macrì map.**

Let S be a smooth projective surface over $\mathbb{C}$. Let $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(S)$ be a stability condition, and $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$ be a choice of Chern character. Assume that we are given a *flat family* [BM14a, Definition 3.1] $\mathcal{E} \in \mathrm{D}^b(M \times S)$ of σ-semistable objects of class ch parametrized by a proper algebraic space M of finite type over $\mathbb{C}$. Denote $N^1(M) = \mathrm{NS}(M)_\mathbb{R}$ as the group of real Cartier divisors modulo numerical equivalence. Write $N_1(M)$ as the group of real 1-cycles modulo numerical equivalence with respect to the intersection paring with Cartier

divisors. The Bayer-Macrì's numerical Cartier divisor class $\ell_{\sigma,\mathcal{E}} \in N^1(M) = \mathrm{Hom}(N_1(M), \mathbb{R})$ is defined as follows: for any projective integral curve $C \subset M$,

$$(3.1) \quad \ell_{\sigma,\mathcal{E}}([C]) = \ell_{\sigma,\mathcal{E}}.C := \Im\left(-\frac{Z\big(\Phi_{\mathcal{E}}(\mathcal{O}_C)\big)}{Z(\mathrm{ch})}\right) = \Im\left(-\frac{Z\big((p_S)_*\mathcal{E}|_{C \times S}\big)}{Z(\mathrm{ch})}\right),$$

where $\Phi_{\mathcal{E}}\colon \mathrm{D}^{\mathrm{b}}(M) \to \mathrm{D}^{\mathrm{b}}(S)$ is the Fourier-Mukai functor with kernel $\mathcal{E}$, and $\mathcal{O}_C$ is the structure sheaf of C.

Theorem 3.1. [BM14a, Theorem 1.1] *The divisor class $\ell_{\sigma,\mathcal{E}}$ is nef on M. In addition, we have $\ell_{\sigma,\mathcal{E}}.C = 0$ if and only if for two general points $c, c' \in C$, the corresponding objects $\mathcal{E}_c, \mathcal{E}_{c'}$ are S-equivalent.*

Here two semistable objects are *S-equivalent* if their Jordan-Hölder filtrations into stable factors of the same phase have identical stable factors.

Definition 3.2. Let C be a Bridgeland chamber with respect to ch. Assume the existence of the moduli space $M_\sigma(\mathrm{ch})$ for $\sigma \in \mathsf{C}$ with a universal family $\mathcal{E}$. Then $M_{\mathsf{C}}(\mathrm{ch}) := M_\sigma(\mathrm{ch})$ is constant for $\sigma \in \mathsf{C}$. Theorem 3.1 yields a map,

$$\begin{aligned}\ell : \overline{\mathsf{C}} &\longrightarrow \mathrm{Nef}(M_{\mathsf{C}}(\mathrm{ch})) \\ \sigma &\mapsto \ell_{\sigma,\mathcal{E}}\end{aligned}$$

which is called the *local Bayer-Macrì map* for the chamber C w.r.t. ch.

For any $\sigma \in \mathrm{Stab}^\dagger(S)$, after a $\widetilde{\mathrm{GL}_2^+}(\mathbb{R})$-action, we assume that $\sigma = \sigma_{\omega,\beta}$, i.e. skyscraper sheaves are stable of phase 1. Denote

$$\mathbf{v} := v(\mathrm{ch}) = \mathrm{ch} \cdot \mathrm{e}^{\frac{1}{2}\ln\mathrm{td}(S)}.$$

The local Bayer-Macrì map is the composition of the following three maps:

$$\mathrm{Stab}^\dagger(S) \xrightarrow{\pi} H^*_{\mathrm{alg}}(S, \mathbb{Q}) \otimes \mathbb{C} \xrightarrow{\mathcal{I}} \mathbf{v}^\perp \xrightarrow{\theta_{\mathsf{C},\mathcal{E}}} N^1(M_{\mathsf{C}}(\mathrm{ch})).$$

- The map π forgets the heart: $\pi(\sigma_{\omega,\beta}) := \mho_{Z_{\omega,\beta}}$ as (2.6).
- For any $\mho \in H^*_{\mathrm{alg}}(S, \mathbb{Q}) \otimes \mathbb{C}$, define $\mathcal{I}(\mho) := \Im\frac{\mho}{-\langle\mho,\mathbf{v}\rangle_S}$. One can check that $\mathcal{I}(\mho) \in \mathbf{v}^\perp$ (this also follows from the Lemma 3.4), where the perpendicular relation is with respect to the Mukai paring:

$$(3.2) \qquad \mathbf{v}^\perp := \{w \in H^*_{\mathrm{alg}}(S, \mathbb{Q}) \otimes \mathbb{R} \mid \langle w, \mathbf{v}\rangle_S = 0\}.$$

- The third map $\theta_{\mathsf{C},\mathcal{E}}$ is the *algebraic Mukai morphism*. More precisely, for a fixed Mukai vector $w \in \mathbf{v}^\perp$, and an integral curve $C \subset M_{\mathsf{C}}(\mathrm{ch})$,

$$\theta_{\mathsf{C},\mathcal{E}}(w).[C] := \langle w, v(\Phi_{\mathcal{E}}(\mathcal{O}_C))\rangle_S.$$

Definition 3.3. Define $w_{\sigma_{\omega,\beta}}(\mathrm{ch}) := -\Im\left(\overline{\langle\mho_Z, \mathbf{v}\rangle_S} \cdot \mho_Z\right)$. We simply write it as $w_{\omega,\beta}$ or w_σ.

Lemma 3.4. *Fix the Chern character $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$. The line bundle class $\ell_{\sigma_{\omega,\beta}} \in N^1(M_{\sigma_{\omega,\beta}}(\mathrm{ch}))$ (if exists) is given by*

$$(3.3) \qquad\qquad \ell_{\sigma_{\omega,\beta}} \stackrel{\mathbb{R}_+}{=\!=\!=} \theta_{\sigma,\mathcal{E}}(w_{\omega,\beta}),$$

where $w_{\omega,\beta} \in \mathbf{v}^\perp$ is given by

$$(3.4) \qquad w_{\omega,\beta} = (\Im Z(\mathrm{ch})) \, \Re \mho_Z - (\Re Z(\mathrm{ch})) \, \Im \mho_Z.$$

Proof. By the definition, $w_{\omega,\beta} = |Z(E)|^2 \mathcal{I}(\mho_Z)$. Applying the Mukai morphism, we get the equation (3.3). Taking the complex conjugate of equation (2.4) we get $\overline{\langle \mho_Z, \mathbf{v} \rangle_S} = \Re Z(\mathrm{ch}) - \sqrt{-1}\Im Z(\mathrm{ch})$. The relation (3.4) thus follows from the definition of $w_{\omega,\beta}$. By the definition of $\mho_Z$, we have $\langle \Re \mho_Z, \mathbf{v} \rangle_S + \sqrt{-1}\langle \Im \mho_Z, \mathbf{v} \rangle_S = \langle \mho_Z, \mathbf{v} \rangle_S = \Re Z(\mathrm{ch}) + \sqrt{-1}\Im Z(\mathrm{ch})$. We then obtain the perpendicular relation $\langle w_\sigma, \mathbf{v} \rangle_S = (\Im Z(\mathrm{ch})) \langle \Re \mho_Z, \mathbf{v} \rangle_S - (\Re Z(\mathrm{ch})) \langle \Im \mho_Z, \mathbf{v} \rangle_S = 0$. $\qquad \square$

The Mukai morphism is the dual version of Donaldson morphism [BM14a, Proposition 4.4, Remark 5.5]. The surjectivity of the Mukai morphism is not known in general. We will compute the image of the local Bayer-Macrì map in Theorem 4.13.

Let $\mathcal{E}$ be a universal family over $M_\sigma(\mathrm{ch})$. Denote $\mathcal{F}$ the dual universal family over $M_{\Phi(\sigma)}(-\mathrm{ch}^*)$. Then $w_{\omega,-\beta}(-\mathrm{ch}^*) \in v(-\mathrm{ch}^*)^\perp$.

Lemma 3.5. *Fix the Chern character* ch. *Let* $\sigma := \sigma_{\omega,\beta}$ *and assume that* $Z_{\omega,\beta}(\mathrm{ch})$ *belongs to the open upper half plane (as Remark 2.12). Then*

$$\ell_\sigma \cong \ell_{\Phi(\sigma)}, \ \ i.e. \ \theta_{\sigma,\mathcal{E}}(w_{\omega,\beta}(\mathrm{ch})) \cong \theta_{\Phi(\sigma),\mathcal{F}}(w_{\omega,-\beta}(-\mathrm{ch}^*)).$$

Proof. This is a consequence of the isomorphism of the moduli spaces

$$M_{\sigma_{\omega,\beta}}(\mathrm{ch}) \cong M_{\sigma_{\omega,-\beta}}(-\mathrm{ch}^*)$$

induced by the duality functor $\Phi(\cdot) = R\mathcal{H}om(\cdot, \mathcal{O}_S)[1]$. $\qquad \square$

3.2. **The global Bayer-Macrì map.** Let σ be in a chamber C. The line bundle $\ell_{\sigma,\mathcal{E}}$ is only defined *locally*, i.e. $\ell_{\sigma,\mathcal{E}} \in N^1(M_\mathsf{C}(\mathrm{ch}))$. If we take another chamber C', we cannot say $\ell_{\sigma,\mathcal{E}} \in N^1(M_{\mathsf{C}'}(\mathrm{ch}))$ directly. We want to associate $\ell_{\sigma,\mathcal{E}}$ the *global* meaning in the following way.

Let $\sigma \in \mathsf{C}$ and $\tau \in \mathsf{C}'$ be two generic numerical stability conditions in different chambers w.r.t. ch. *Assume $M_\sigma(\mathrm{ch})$ and $M_\tau(\mathrm{ch})$ are non-empty and irreducible with universal families $\mathcal{E}$ and $\mathcal{F}$ respectively. And assume that there is a birational map between $M_\sigma(\mathrm{ch})$ and $M_\tau(\mathrm{ch})$, induced by a derived autoequivalence Ψ of $\mathrm{D}^\mathrm{b}(S)$ in the following sense: there exists a common open subset U of $M_\sigma(\mathrm{ch})$ and $M_\tau(\mathrm{ch})$, with complements of codimension at least two,* such that for any $u \in U$, the corresponding objects $\mathcal{E}_u \in M_\sigma(\mathrm{ch})$ and $\mathcal{F}_u \in M_\tau(\mathrm{ch})$ are related by $\mathcal{F}_u = \Psi(\mathcal{E}_u)$. Then the Néron-Severi groups of $M_\sigma(\mathrm{ch})$ and $M_\tau(\mathrm{ch})$ can canonically be identified. So for a Mukai vector $w \in \mathbf{v}^\perp$, the two line bundles $\theta_{\mathsf{C},\mathcal{E}}(w)$ and $\theta_{\mathsf{C}',\mathcal{F}}(w)$ are identified.

Definition 3.6. Fix a base geometric numerical stability condition σ in a chamber. A *global* Bayer-Macrì map

$$\ell : \mathrm{Stab}^\dagger(S) \to N^1(M_\sigma(\mathrm{ch})),$$

is glued by the *local* Bayer-Macrì map by the above identification.

Theorem 3.7 ([BM14b] for K3,[LZ16] for $\mathbb{P}^2$). *Let S be a K3 surface or the projective plane $\mathbb{P}^2$. Let* ch *be primitive character over S. There is a global Bayer-Macrì map.*

4. Bayer-Macrì decomposition

In this section, we give an *intrinsic* decomposition of the Mukai vector $w_{\omega,\beta}(\mathrm{ch})$ in Lemma 4.5 and Lemma 4.8 respectively, according to the the dimension of support of objects with invariants ch. In particular, each component is in $\mathbf{v}^{\perp}$. So we can apply $\theta_{\sigma,\mathcal{E}}$ and obtain the *intrinsic* decomposition of $\ell_{\sigma_{\omega,\beta}}$. We call such decomposition of $w_{\omega,\beta}$ or $\ell_{\sigma_{\omega,\beta}}$ as the *Bayer-Macrì decomposition*.

4.1. **Preliminary computation by using $\mho_Z$.**

Lemma 4.1. *If $\Im Z(\mathrm{ch}) = 0$, then $w_{\sigma} \stackrel{\mathbb{R}_+}{=\!=\!=} \Im \mho_Z$. If $\Im Z(\mathrm{ch}) > 0$, then*

$$
\begin{aligned}
(4.1) \qquad w_{\sigma} \stackrel{\mathbb{R}_+}{=\!=\!=} \;& \mu_{\sigma}(\mathrm{ch})\Im\mho_Z + \Re\mho_Z \\
= \;& \left(0, \mu_{\sigma}(\mathrm{ch})\omega + \beta, -\frac{3}{4}\mathrm{K}_S.(\mu_{\sigma}(\mathrm{ch})\omega + \beta) \right) \\
& + \left(1, -\frac{3}{4}\mathrm{K}_S, -\frac{1}{2}\chi(\mathcal{O}_S) + \frac{11}{32}\mathrm{K}_S^2 \right) \\
& + \left(0, 0, \beta.(\mu_{\sigma}(\mathrm{ch})\omega + \beta) - \frac{1}{2}(\omega^2 + \beta^2) \right).
\end{aligned}
$$

Proof. The case for $\Im Z(\mathrm{ch}) = 0$ follows from equation (3.4). If $\Im Z(\mathrm{ch}) > 0$, we divide equation (3.4) by this positive number and obtain equation (4.1). The concrete formula is then derived by equation (2.5). $\qquad\square$

Lemma 4.2. *Fix a frame (H, γ, u). We have relations:*

$$
(4.2) \qquad\qquad\qquad \mu_{\sigma}(\mathrm{ch})\omega + \beta = C(\mathrm{ch}, \mathrm{ch}')H + u\gamma,
$$

$$
(4.3) \qquad \beta.(\mu_{\sigma}(\mathrm{ch})\omega + \beta) - \frac{1}{2}(\omega^2 + \beta^2) = -\frac{g}{2}D(\mathrm{ch}, \mathrm{ch}') - \frac{d}{2}u^2,
$$

where the numbers $C(\mathrm{ch}, \mathrm{ch}')$ and $D(\mathrm{ch}, \mathrm{ch}')$ are given by equations (2.9, 2.10).

Proof. The proof is a direct computation by using Maciocia's Theorem 2.4. For the reader's convenience, we give the details. For the equation (4.2), we only need to check that

$$
\mu_{\sigma}(\mathrm{ch})t + s = C(\mathrm{ch}, \mathrm{ch}').
$$

Recall the wall equation is $(s - C)^2 + t^2 = D + C^2$. Now

$$
\begin{aligned}
\mu_{\sigma}(\mathrm{ch})t + s &= \frac{z - sy_1 g + uy_2 d + \frac{x}{2}\left(s^2 g - u^2 d - t^2 g\right)}{(y_1 - xs)g} + s \\
&= \frac{z + uy_2 d - \frac{x}{2}u^2 d - \frac{xg}{2}(s^2 + t^2)}{(y_1 - xs)g} \\
&= \frac{z + uy_2 d - \frac{x}{2}u^2 d - \frac{xg}{2}(2sC + D)}{(y_1 - xs)g} \qquad \text{by using wall equation.}
\end{aligned}
$$

So we only need to check that

$$
(4.4) \qquad z + uy_2 d - \frac{x}{2}u^2 d - \frac{xg}{2}(2sC + D) = (y_1 - xs)gC.
$$

If $x = 0$, the equation (4.4) is true since $C = \frac{z+duy_2}{gy_1}$. If $x \neq 0$, the equation (4.4) is still true by using equation (2.11).

Let us prove the equation (4.3).

$$
\begin{aligned}
\text{LHS of (4.3)} &= (sH + u\gamma).(CH + u\gamma) - \frac{g}{2}(t^2 + s^2 - \frac{d}{g}u^2) \\
&= sCg - u^2d - \frac{g}{2}(2sC + D - \frac{d}{g}u^2) = \text{RHS of (4.3)}
\end{aligned}
$$

$\square$

Definition 4.3. Fix a frame (H, γ, u). Define the vector $\mathbf{t}_{(H,\gamma,u)}(\text{ch}, \text{ch}')$ as

$$
\left(1, CH + u\gamma - \frac{3}{4}\mathrm{K}_S, -\frac{3}{4}\mathrm{K}_S.(CH + u\gamma) - \frac{1}{2}\chi(\mathcal{O}_S) + \frac{11}{32}\mathrm{K}_S^2 \right),
$$

where the center $C = C(\text{ch}, \text{ch}')$ is as the equation (2.9).

Lemma 4.4. *If* $\Im Z_{\omega,\beta}(\text{ch}) > 0$, *then*

$$
(4.5) \quad w_{\sigma \in W(\text{ch},\text{ch}')} \xlongequal{\mathbb{R}_+} \left(\frac{g}{2}D(\text{ch}, \text{ch}') + \frac{d}{2}u^2 \right)(0, 0, -1) + \mathbf{t}_{(H,\gamma,u)}(\text{ch}, \text{ch}').
$$

Proof. This is a direct computation by using equations (4.1, 4.2, 4.3). $\square$

4.2. **The local Bayer-Macrì decomposition.** We decompose w_σ in three cases, according to the the dimension of support of objects with invariants ch. *Assume* there is a flat family $\mathcal{E} \in D^b(M_\sigma(\text{ch}) \times S)$ and denote the Mukai morphism by $\theta_{\sigma,\mathcal{E}}$.

4.2.1. *Supported in dimension 0.* Fix $\text{ch} = (0, 0, n)$, with n a positive integer. Fix a frame (H, γ, u). Since $t > 0$ is the trivial chamber and there is no wall on $\Pi_{(H,\gamma,u)}$, we obtain $w_\sigma \xlongequal{\mathbb{R}_+} \Im \mathcal{U}_Z \xlongequal{\mathbb{R}_+} \left(0, H, (\beta - \frac{3}{4}\mathrm{K}_S).H \right)$, and the nef line bundle $\ell_\sigma = \theta_{\sigma,\mathcal{E}}(0, H, (sH - \frac{3}{4}\mathrm{K}_S).H)$ on the moduli space $M_\sigma(\text{ch}) \cong \mathrm{Sym}^n(S)$ [LQ14, Lemma 2.10], which is independent of s.

4.2.2. *Supported in dimension 1.* Fix a frame (H, γ, u). We assume that $\text{ch} = (0, \text{ch}_1, \text{ch}_2)$ with $\text{ch}_1.H > 0$. Now the center is given by $C = \frac{z+duy_2}{gy_1}$, which is independent of ch'. So the vector

$$
\mathbf{t}_{(H,\gamma,u)}(\text{ch}) := \mathbf{t}_{(H,\gamma,u)}(\text{ch}, \text{ch}')
$$

is also independent of ch'. There is another special vector

$$
w_{\infty H, \beta} \xlongequal{\mathbb{R}_+} (0, 0, -1).
$$

We get two well-defined line bundles in the following theorem:

$$
(4.6) \qquad \mathcal{S} := \theta_{\sigma,\mathcal{E}}(0, 0, -1), \quad \mathcal{T}_{(H,\gamma,u)}(\text{ch}) := -\theta_{\sigma,\mathcal{E}}(\mathbf{t}_{(H,\gamma,u)}(\text{ch})).
$$

Lemma 4.5. *(The local Bayer-Macrì decomposition in dimension* 1*.) Fix a frame* (H, γ, u). *Assume* $\text{ch} = (0, \text{ch}_1, \text{ch}_2)$ *with* $\text{ch}_1.H > 0$.

(a). *There is a decomposition*

$$
w_{\sigma \in W(\text{ch},\text{ch}')} \xlongequal{\mathbb{R}_+} \left(\frac{g}{2}D(\text{ch}, \text{ch}') + \frac{d}{2}u^2 \right)(0, 0, -1) + \mathbf{t}_{(H,\gamma,u)}(\text{ch}),
$$

where $(0, 0, -1)$, $\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}) \in \mathbf{v}^{\perp}$. *Moreover* $r = \mathrm{ch}'_0 \neq 0$ *and the coefficient before* $(0, 0, -1)$ *is expressed in terms of potential destabilizing Chern character* $\mathrm{ch}' = (r, c_1 H + c_2 \gamma + \delta', \chi)$:

$$(4.7) \qquad \frac{g}{2} D(\mathrm{ch}, \mathrm{ch}') + \frac{d}{2} u^2 = \frac{\chi - gCc_1 + udc_2}{r}.$$

(b). *Assume that there is a flat family* $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(M_\sigma(\mathrm{ch}) \times S)$. *Then the Bayer-Macrì nef line bundle on the moduli space* $M_\sigma(\mathrm{ch})$ *has a decomposition*

$$(4.8) \qquad \ell_{\sigma \in W(\mathrm{ch},\mathrm{ch}')} \overset{\mathbb{R}_+}{=\!=\!=} \left(\frac{g}{2} D(\mathrm{ch}, \mathrm{ch}') + \frac{d}{2} u^2 \right) \mathcal{S} - \mathcal{T}_{(H,\gamma,u)}(\mathrm{ch}).$$

Proof. Part (a) follows from computation. The Mukai vector $\mathbf{v}$ is given by

$$\mathbf{v} = (0, \mathrm{ch}_1, \mathrm{ch}_2 - \frac{1}{4} \mathrm{ch}_1 . \mathrm{K}_S).$$

So $(0, 0, -1) \in \mathbf{v}^{\perp}$ by the definition equation (3.2) and the formula (2.3). To show $\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}) \in \mathbf{v}^{\perp}$, we can either directly compute the Mukai pairing:

$$\langle \mathbf{t}_{(H,\gamma,u)}(\mathrm{ch}), \mathbf{v} \rangle_S = (CH + u\gamma - \frac{3}{4} \mathrm{K}_S) \cdot \mathrm{ch}_1 - (\mathrm{ch}_2 - \frac{1}{4} \mathrm{ch}_1 . \mathrm{K}_S - \frac{1}{2} \mathrm{ch}_1 . \mathrm{K}_S) = 0,$$

or notice the relation (4.5) and the fact $w_{\sigma \in W(\mathrm{ch},\mathrm{ch}')} \in \mathbf{v}^{\perp}$, $(0, 0, -1) \in \mathbf{v}^{\perp}$. Then $\mathcal{S}$ and $\mathcal{T}_{(H,\gamma,u)}(\mathrm{ch})$ are well defined in (4.6). Recall the equation (2.10) for $D(\mathrm{ch}, \mathrm{ch}')$. Since $x = \mathrm{ch}_0 = 0$, we obtain $r \neq 0$. The relation (4.7) is then derived by using equation (2.14). Part (b) follows from Part (a) by applying the Mukai morphism $\theta_{\sigma,\mathcal{E}}$. $\square$

4.2.3. *Supported in dimension 2.* Assume that $\mathrm{ch}_0 \neq 0$. If $\mathrm{ch}_0 < 0$, we observe

$$w_{\infty H,\beta} \overset{\mathbb{R}_+}{=\!=\!=} (0, H, (\frac{\mathrm{ch}_1}{\mathrm{ch}_0} - \frac{3}{4} \mathrm{K}_S) H) \overset{\mathbb{R}_+}{=\!=\!=} w_{\sigma \in \mathtt{DUW}}.$$

Definition 4.6. Fix $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$ with $\mathrm{ch}_0 \neq 0$, define

$$\mathbf{w}(\mathrm{ch}) \ := \ \left(1, -\frac{3}{4} \mathrm{K}_S, -\frac{\mathrm{ch}_2}{\mathrm{ch}_0} - \frac{1}{2} \chi(\mathcal{O}_S) + \frac{11}{32} \mathrm{K}_S^2 \right),$$

$$\mathbf{m}(L, \mathrm{ch}) \ := \ \left(0, L, (\frac{\mathrm{ch}_1}{\mathrm{ch}_0} - \frac{3}{4} \mathrm{K}_S) . L \right), \ \text{where } L \in N^1(S),$$

$$\mathbf{u}(\mathrm{ch}) := \mathbf{w}(\mathrm{ch}) + \mathbf{m}(\frac{1}{2} \mathrm{K}_S, \mathrm{ch}) = \left(1, -\frac{1}{4} \mathrm{K}_S, -\frac{\mathrm{ch}_2}{\mathrm{ch}_0} + \frac{\mathrm{ch}_1 . \mathrm{K}_S}{2 \mathrm{ch}_0} - \frac{1}{2} \chi(\mathcal{O}_S) - \frac{1}{32} \mathrm{K}_S^2 \right).$$

Lemma 4.7. *We have the following three perpendicular relations for Mukai vectors:*

$$\mathbf{m}(L, \mathrm{ch}), \ \mathbf{w}(\mathrm{ch}), \ \mathbf{u}(\mathrm{ch}) \in \mathbf{v}^{\perp}.$$

Proof. The perpendicular relations can be checked directly by definition equation (3.2) and equations (2.2, 2.3). $\square$

Lemma 4.8. *(The local Bayer-Macrì decomposition in dimension 2.)*

(a). *If* $\mathrm{ch}_0 \neq 0$ *and* $\Im Z(\mathrm{ch}) > 0$, *then there is a decomposition (up to a positive scalar):*

$$(4.9) \qquad w_{\omega,\beta}(\mathrm{ch}) \xlongequal{\mathbb{R}_+} \mu_\sigma(\mathrm{ch})\mathbf{m}(\omega,\mathrm{ch}) + \mathbf{m}(\beta,\mathrm{ch}) + \mathbf{w}(\mathrm{ch})$$

$$(4.10) \qquad\qquad = \mu_\sigma(\mathrm{ch})\mathbf{m}(\omega,\mathrm{ch}) + \mathbf{m}(\alpha,\mathrm{ch}) + \mathbf{u}(\mathrm{ch}),$$

where $\mathbf{m}(\omega,\mathrm{ch})$, $\mathbf{m}(\beta,\mathrm{ch})$, $\mathbf{m}(\alpha,\mathrm{ch})$, $\mathbf{w}(\mathrm{ch})$, $\mathbf{u}(\mathrm{ch}) \in \mathbf{v}^\perp$.

(b). *Assume there is a flat family $\mathcal{E}$. Then the Bayer-Macrì line bundle class $\ell_{\sigma_{\omega,\beta}}$ has a decomposition in $N^1(M_\sigma(\mathrm{ch}))$:*

$$(4.11) \qquad \ell_{\sigma_{\omega,\beta}} \xlongequal{\mathbb{R}_+} \mu_\sigma(\mathrm{ch})\theta_{\sigma,\mathcal{E}}(\mathbf{m}(\omega,\mathrm{ch})) + \theta_{\sigma,\mathcal{E}}(\mathbf{m}(\beta,\mathrm{ch})) + \theta_{\sigma,\mathcal{E}}(\mathbf{w}(\mathrm{ch})).$$

Proof. Recall the equation (4.1). To show the relation (4.9), we only need to check that

$$(4.12) \qquad \frac{\mathrm{ch}_1}{\mathrm{ch}_0}.(\mu_\sigma(\mathrm{ch})\omega + \beta) - \frac{\mathrm{ch}_2}{\mathrm{ch}_0} = \beta.(\mu_\sigma(\mathrm{ch})\omega + \beta) - \frac{1}{2}(\omega^2 + \beta^2).$$

By definition of Bridgeland slope, we have

$$\mu_\sigma(\mathrm{ch}) = \frac{\mathrm{ch}_2 - \frac{1}{2}\mathrm{ch}_0\left(\omega^2 - \beta^2\right) - \mathrm{ch}_1.\beta}{\omega.\left(\mathrm{ch}_1 - \mathrm{ch}_0\beta\right)}.$$

So

$$\mu_\sigma(\mathrm{ch})\omega.\left(\frac{\mathrm{ch}_1}{\mathrm{ch}_0} - \beta\right) = \frac{\mathrm{ch}_2}{\mathrm{ch}_0} - \frac{1}{2}\left(\omega^2 - \beta^2\right) - \frac{\mathrm{ch}_1}{\mathrm{ch}_0}.\beta.$$

Therefore we have the relation (4.12). The equation (4.10) follows from equation (4.9) and the relation $\alpha = \beta - \frac{1}{2}\mathrm{K}_S$. Part (b) follows directly by applying the Mukai morphism $\theta_{\sigma,\mathcal{E}}$. □

Remark 4.9. An equivalent decomposition of $w_{\omega,\beta}(\mathrm{ch})$ (4.9) is also obtained by Bolognese, Huizenga, Lin, Riedl, Schmidt, Woolf, Zhao [BHL+16, Proposition 3.8.].

4.3. **The global Bayer-Macrì decomposition.** Assume the existence of the global Bayer-Macrì map. Assume that $W(\mathrm{ch}, \mathrm{ch}')$ is an actual Bridgeland wall.

4.3.1. *Supported in dimension 1.*

Theorem 4.10. *Fix a frame (H, γ, u) and assume $\mathrm{ch} = (0, \mathrm{ch}_1, \mathrm{ch}_2)$ with $\mathrm{ch}_1.H > 0$. Assume the existence of the global Bayer-Macrì map, with the fixed base stability condition in the Simpson chamber* SC, *i.e.* $M_{\sigma \in \mathtt{SC}}(\mathrm{ch}) \cong M_{(\alpha,\omega)}(\mathrm{ch})$. *Then there is a correspondence from Bridgeland wall $W(\mathrm{ch}, \mathrm{ch}')$ as semicircle (2.8) in the half-plane $\Pi_{(H,\gamma,u)}$ with fixed center C (or equivalently, as semi-line (2.20) in the plane $\Sigma_{(H,\gamma,u)}$ with fixed slope C) to the Mori wall inside pseudo-effective cone $\overline{\mathrm{Eff}}(M_{(\alpha,\omega)})(\mathrm{ch})$:*

$$(4.13) \qquad \ell_{\sigma \in W(\mathrm{ch},\mathrm{ch}')} \xlongequal{\mathbb{R}_+} \left(\frac{g}{2}D(\mathrm{ch},\mathrm{ch}') + \frac{d}{2}u^2\right)\mathcal{S} - \mathcal{T}_{(H,\gamma,u)}(\mathrm{ch}),$$

$$(4.14) \qquad\qquad = \frac{\chi - gCc_1 + udc_2}{r}\mathcal{S} - \mathcal{T}_{(H,\gamma,u)}(\mathrm{ch}).$$

Proof. The center $C = \frac{z+duy_2}{gy_1}$ is independent of ch′. So is $\mathcal{T}_{(H,\gamma,u)}(\mathrm{ch})$. The number $D(\mathrm{ch}, \mathrm{ch}')$ is given by equation (2.15). Since the base stability condition $\sigma \in \mathtt{SC}$, we obtain

$$\mathcal{S} = \theta_{\mathtt{SC},\mathcal{E}}((0,0,-1)) \in N^1(M_{(\alpha,\omega)}(\mathrm{ch})),$$

$$\mathcal{T}_{(H,\gamma,u)}(\mathrm{ch}) = -\theta_{\mathtt{SC},\mathcal{E}}(\mathbf{t}_{(H,\gamma,u)}(\mathrm{ch})) \in N^1(M_{(\alpha,\omega)}(\mathrm{ch})).$$

The equation (4.13) follows from equation (4.8) by fixing above two line bundles $\mathcal{S}$ and $\mathcal{T}_{(H,\gamma,u)}(\mathrm{ch})$ in the Simpson moduli space. We obtain equation (4.14) by using equation (4.7). □

By using the Donaldson morphism $\lambda_\mathcal{E}$, we have [HL10, Example 8.1.3]

$$\mathcal{S} = \lambda_\mathcal{E}(0,0,1) = (p_M)_*(\det(\mathcal{E})|_{M\times\{s\}}).$$

The line bundle $\mathcal{S}$ is conjectured to induce the support morphism, which maps $E \in M_{\sigma\in\mathtt{SC}}(\mathrm{ch})$ to $\mathrm{Supp}(E)$. This is proved for the case that $S = \mathbb{P}^2$ [Woo13] or S is a K3 surface [BM14b, Lemma 11.3].

4.3.2. *Supported in dimension 2.* By applying the derived dual functor if necessary, we further assume $\mathrm{ch}_0 > 0$ in Lemma 4.8. Recall Lemma 3.5 and Corollary 2.11. We obtain

$$w_{\omega,-\beta}(-\mathrm{ch}^*) \overset{\mathbb{R}_+}{=\!=\!=} \mu_{\Phi(\sigma)}(-\mathrm{ch}^*)\mathbf{m}(\omega, -\mathrm{ch}^*) + \mathbf{m}(-\beta, \mathrm{ch}) + \mathbf{w}(-\mathrm{ch}^*),$$

$$\ell_\sigma \cong \ell_{\Phi(\sigma)} \overset{\mathbb{R}_+}{=\!=\!=} \mu_{\Phi(\sigma)}(-\mathrm{ch}^*)\theta_{\Phi(\sigma),\mathcal{F}}(\mathbf{m}(\omega, -\mathrm{ch}^*))$$
$$+\theta_{\Phi(\sigma),\mathcal{F}}(\mathbf{m}(-\beta, -\mathrm{ch}^*)) + \theta_{\Phi(\sigma),\mathcal{F}}(\mathbf{w}(-\mathrm{ch}^*)).$$

Since $\mu_{\Phi(\sigma)}(-\mathrm{ch}^*) = -\mu_\sigma(\mathrm{ch})$, we get

$$\theta_{\Phi(\sigma),\mathcal{F}}(\mathbf{w}(-\mathrm{ch}^*)) \cong \theta_{\sigma,\mathcal{E}}(\mathbf{w}(\mathrm{ch})), \quad \theta_{\Phi(\sigma),\mathcal{F}}(\mathbf{m}(\omega, -\mathrm{ch}^*)) \cong -\theta_{\sigma,\mathcal{E}}(\mathbf{m}(\omega, \mathrm{ch})).$$

Notation 4.11. Assume $\mathrm{ch}_0 > 0$. Let L be a line bundle on S. Denote

$$\widetilde{L} := \theta_{\Phi(\sigma),\mathcal{F}}(\mathbf{m}(L, -\mathrm{ch}^*)) \cong -\theta_{\sigma,\mathcal{E}}(\mathbf{m}(L, \mathrm{ch})), \quad \mathcal{B}_0 := -\theta_{\sigma,\mathcal{E}}(\mathbf{u}(\mathrm{ch})).$$

Then we have

(4.15) $$\theta_{\sigma,\mathcal{E}}(\mathbf{w}(\mathrm{ch})) = \frac{1}{2}\widetilde{\mathrm{K}_S} - \mathcal{B}_0.$$

Recall $\alpha = \beta - \frac{1}{2}\mathrm{K}_S$. Denote

$$\mathcal{B}_\alpha := \widetilde{\beta} - \theta_{\Phi(\sigma),\mathcal{F}}(\mathbf{w}(-\mathrm{ch}^*)) \cong \widetilde{\beta} - \theta_{\sigma,\mathcal{E}}(\mathbf{w}(\mathrm{ch})) = \widetilde{\alpha} + \mathcal{B}_0.$$

$\widetilde{L}$, $\mathcal{B}_\alpha$ and $\mathcal{B}_0$ are line bundles on $M_\sigma(\mathrm{ch})$.

Assumption 4.12. An Chern character $\mathrm{ch} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$ satisfies *condition* (C) if the following three assumptions holds:

- $\mathrm{ch}_0 > 0$;
- (Bogomolov type) $\mathrm{ch}_1^2 - 2\mathrm{ch}_0\mathrm{ch}_2 \geq 0$;
- $\gcd(\mathrm{ch}_0, \mathrm{ch}_1.H, \mathrm{ch}_2 - \frac{1}{2}\mathrm{ch}_1.\mathrm{K}_S) = 1$ for a fixed ample line bundle H ([HL10, Corollary 4.6.7]).

Theorem 4.13. *Fix* ch, *and assume it satisfies condition* (C). *Assume the existence of the global Bayer-Macrì map, with the fixed base stability condition in the Gieseker chamber* GC, *i.e.* $M_{\sigma\in\mathtt{GC}}(\mathrm{ch}) \cong M_{(\alpha,\omega)}(\mathrm{ch})$. *Then the following conclusions hold.*

(a). There is a Bayer-Macrì decomposition for the line bundle $\ell_{\sigma_{\omega,\beta}}$:

$$\ell_{\sigma_{\omega,\beta}} \overset{\mathbb{R}_+}{=\!=\!=} (-\mu_{\sigma_{\omega,\beta}}(\mathrm{ch}))\widetilde{\omega} - \mathcal{B}_\alpha = (-\mu_{\sigma_{\omega,\beta}}(\mathrm{ch}))\widetilde{\omega} - \widetilde{\alpha} - \mathcal{B}_0.$$

(b). The line bundle $\widetilde{\omega}$ induces the Gieseker-Uhlenbeck morphism from the (α,ω)-Gieseker semistable moduli space $M_{(\alpha,\omega)}(\mathrm{ch})$ to the Uhlenbeck space $U_\omega(\mathrm{ch})$.

(c). If $\mathrm{ch}_0 = 2$ and $\partial M_{(\alpha,\omega)}(\mathrm{ch}) \neq \emptyset$, the divisor $\mathcal{B}_\alpha$ is the α-twisted boundary divisor of the induced Gieseker-Uhlenbeck morphism. In particular, in the case of $\alpha = 0$, the divisor $\mathcal{B}_0$ is the (untwisted) boundary divisor from the ω-semistable Gieseker moduli space $M_\omega(\mathrm{ch})$ to the Uhlenbeck space $U_\omega(\mathrm{ch})$.

(d). Fix a frame (H, γ, u). Then there is a correspondence from the Bridgeland wall $W(\mathrm{ch}, \mathrm{ch}')$ as semicircle (2.8) in the half-plane $\Pi_{(H,\gamma,u)}$ with center $C(\mathrm{ch}, \mathrm{ch}')$ (or equivalently, as semi-line (2.19) in the plane $\Sigma_{(H,\gamma,u)}$ with slope $C(\mathrm{ch}, \mathrm{ch}')$) to the effective line bundle on the moduli space $M_{(\alpha,\omega)}(\mathrm{ch})$:

$$(4.16) \qquad \ell_{\sigma \in W(\mathrm{ch},\mathrm{ch}')} \overset{\mathbb{R}_+}{=\!=\!=} -C(\mathrm{ch}, \mathrm{ch}')\widetilde{H} - u\widetilde{\gamma} + \frac{1}{2}\widetilde{\mathrm{K}_S} - \mathcal{B}_0.$$

Proof. We identify $\widetilde{\omega}$ and $\mathcal{B}_\alpha$ as line bundles on $M_{(\alpha,\omega)}(\mathrm{ch})$. The equation (4.11) implies Part (a). Since $-\mu_{\sigma \in \mathrm{UW}}(\mathrm{ch}) = \mu_{\Phi(\sigma) \in \mathrm{DUW}}(-\mathrm{ch}^*) = +\infty$, we obtain

$$\ell_{\sigma \in \mathrm{UW}} \overset{\mathbb{R}_+}{=\!=\!=} \widetilde{\omega}.$$

Part (b) follows from [HL10, Theorem 8.2.8]. If $\mathrm{ch}_0 = 2$ and $\partial M_{(\alpha,\omega)}(\mathrm{ch}) \neq \emptyset$, the Gieseker-Uhlenbeck morphism is a divisorial contraction by [HL10, Lemma 9.2.1], and $\mathcal{B}_\alpha$ is the boundary divisor from the α-twisted moduli space $M_{(\alpha,\omega)}(\mathrm{ch})$ to the Uhlenbeck space $U_\omega(\mathrm{ch})$. In particular, the $\mathcal{B}_0$ is the untwisted boundary divisor. This shows Part (c). The equation (4.16) follows from a direct computation by by using equations (4.11, 4.2, 4.15):

$$\begin{aligned} \ell_{\sigma \in W(\mathrm{ch},\mathrm{ch}')} &\overset{\mathbb{R}_+}{=\!=\!=} \theta_{\sigma,\mathcal{E}}(\mathbf{m}(\mu_\sigma(\mathrm{ch})\omega + \beta, \mathrm{ch})) + \theta_{\sigma,\mathcal{E}}(\mathbf{w}(\mathrm{ch})) \\ &= -C(\mathrm{ch}, \mathrm{ch}')\widetilde{H} - u\widetilde{\gamma} + \frac{1}{2}\widetilde{\mathrm{K}_S} - \mathcal{B}_0. \end{aligned}$$

$\square$

There are two line bundles $\mathcal{L}_0$, $\mathcal{L}_1$ on Gieseker moduli space introduced by Le Potier. We follow notations as in [HL10, Definition 8.1.9.]. Then

$$\mathcal{L}_0 = -\mathrm{ch}_0 \mathcal{B}_0, \quad \mathcal{L}_1 = \mathrm{ch}_0 \widetilde{H}.$$

Arcara, Bertram, Coskun and Huizenga [ABCH13] studied the Hilbert scheme of n-points on the projective plane $\mathbb{P}^2$ and gave a precise conjecture between the Bridgeland walls and Mori walls, which was one of the motivation of Bayer-Macrì's line bundle theory. This conjecture is proved by Li and Zhao [LZ13]. The relation still holds for more general primitive character [LZ16].

Corollary 4.14 ([LZ16])**.** *Let $S = \mathbb{P}^2$ and denote H the hyperplane divisor on $\mathbb{P}^2$. Fix ch primitive with $\mathrm{ch}_0 > 0$. Assume $M_H(\mathrm{ch}) \neq \emptyset$. Then there is*

a relation

$$\ell_{\sigma \in W(\mathrm{ch},\mathrm{ch}')} \xlongequal{\mathbb{R}_+} -\left(C(\mathrm{ch},\mathrm{ch}') + \frac{3}{2}\right)\widetilde{H} - \mathcal{B}_0.$$

Proof. The existence of the global Bayer-Macrì map is proved by Li and Zhao [LZ16, Theorem 0.2]. We then apply equation (4.16) with $\gamma = 0$, $K_S = -3H$. $\square$

Example 4.15. Assume that the irregularity of the surface is 0. If $\mathrm{ch} = (1,0,-n)$, then the Gieseker-Uhlenbeck morphism is the Hilbert-Chow morphism $h : S^{[n]} \to S^{(n)}$, which maps the Hilbert scheme of n-points on S to the symmetric product $S^{(n)}$. In particular,

$$\widetilde{H} = \mathcal{L}_1 = h^*(\mathcal{O}_{S^{(n)}}(1)),$$

which induces the Hilbert-Chow morphism [HL10, Example 8.2.9]. The boundary divisor of Hilbert-Chow morphism is

(4.17) $$\mathrm{B} := \{\xi \in S^{[n]} : |\mathrm{Supp}(\xi)| < n\}.$$

It is known from the Appendix in [BSG91] that $\frac{1}{2}\mathrm{B}$ is an integral divisor and $\mathcal{B}_0 = -\mathcal{L}_0 = \frac{1}{2}\mathrm{B}$.

5. A TOY MODEL: FIBERED SURFACE OVER $\mathbb{P}^1$ WITH A GLOBAL SECTION

We compute the nef cone of the Hilbert scheme $S^{[n]}$ of n-points over S by using the Theorem 4.13. Here let $\pi : S \to \mathbb{P}^1$ be either a $\mathbb{P}^1$-fibered or an elliptic-fibered surface over $\mathbb{P}^1$ with a global section E whose self-intersection number is $-e$. We *assume* that all fibers are reduced and irreducible, and the Picard group of S is generated by E and F, where F is the generic fiber class. We have the intersection numbers:

$$E.E = -e, \quad E.F = 1, \quad F^2 = 0.$$

- $\mathbb{P}^1$ fibration. In this case, $F \cong \mathbb{P}^1$ and S is the *Hirzebruch surface* Σ_e with integer $e \geq 0$. Then $K_S = -2(E + eF) + (e-2)F$. Here Σ_0 is the surface $\mathbb{P}^1 \times \mathbb{P}^1$.
- Elliptic fibration. In this case, the generic fiber F is an elliptic curve. We denote the surface by S_e and further *assume* that $e \geq 2$. Then S_e has the unique section E and $K_S = (e-2)F$ [Mir89].

Since the nef cone of S is generated by the two extremal nef line bundle $E + eF$ and F, any ample line bundle H, after rescaling, can be written as

$$H := \lambda(E + eF) + (1-\lambda)F, \quad 0 < \lambda < 1.$$

Take γ such that $H.\gamma = 0$, $H^2 = -\gamma^2$. Basic computation shows that $\gamma = \pm(-\lambda(E + eF) + (1 - \lambda + e\lambda)F)$. An (H,γ,u)-frame, with $u \geq 0$, is fixed by the choice:

$$\gamma := -\lambda(E + eF) + (1 - \lambda + e\lambda)F.$$

The two numbers λ and u are regarded as the *initial values*.

Fix $\mathrm{ch} = (1,0,-n)$, with integer $n \geq 2$. The potential walls are given by $(s - C)^2 + t^2 = C^2 + D$ with $t > 0$ where

$$C = C(\mathrm{ch},\mathrm{ch}') = \frac{\mathrm{ch}_2' + \mathrm{ch}_0'n - u\mathrm{ch}_1'.\gamma}{\mathrm{ch}_1'.H}, \quad D = D(\mathrm{ch},\mathrm{ch}') = -u^2 - \frac{2n}{H^2}.$$

Recall $s_0 := \frac{ch_1.H}{ch_0 H^2} = 0$. The UW is given by $s = s_0 = 0$. Therefore $C < 0$.

One type of nef line bundle on $S^{[n]}$ is $\widetilde{\omega}$ for $\omega \in \mathrm{Amp}(S)$, which induces Gieseker-Uhlenbeck morphism. By taking ω to be extremal, i.e $\omega = E + eF$ or F, we obtain two extremal nef line bundles on $S^{[n]}$:

$$(5.1) \qquad\qquad (\widetilde{E + eF}), \quad \widetilde{F}.$$

To find the nef cone of $S^{[n]}$, we need to find the biggest non-trivial wall, i.e the smallest value of C. Let us call such wall as *Gieseker wall*.

Lemma 5.1. *If Gieseker wall is given by rank one wall, then*

$$\mathrm{ch}' = (1, -F, 0) \quad or \quad \mathrm{ch}' = (1, -E, \frac{-e}{2}).$$

Proof. The idea is the same as [ABCH13] (see also [BC13]). Any destabilizing subsheaf of I_Z of rank one has the form $L \otimes I_W$ with Chern character $\mathrm{ch}' = (1, L, \frac{L^2}{2} - w)$ for some line bundle L and some ideal sheaf I_W of length $w \geq 0$. Then $C(\mathrm{ch}, \mathrm{ch}') = \frac{\frac{L^2}{2} + n - uL.\gamma}{L.H} + \frac{w}{-L.H}$. To guarantee $L \otimes I_W$ as an object in the heart, we need $L.H < 0$. Then we write $L = -(mF + kE)$, with two non-negative integers m and k, and $(m, k) \neq (0, 0)$. To get the biggest non-trivial wall, we must take $w = 0$. Denote the line bundle on $S^{[n]}$ corresponding to the destabilizing line bundle $-(mF + kE)$ by $\ell(m, k)$. The locus contracted by $\ell(0, 1)$ is $\{Z \in S^{[n]} | Z \subset E\}$. The locus contracted by $\ell(1, 0)$ is $\{Z \in S^{[n]} | Z \subset F, Z \text{ is linear equivalent to } n(E \cap F)\}$. Assume that the smallest value is obtained by taking $(m, k) \neq (1, 0)$ nor $(0, 1)$. Recall the walls are nested. But the locus contracted by $\ell(1, 0)$ or $\ell(0, 1)$ are also contracted by $\ell(m, k)$, which is a contradiction. $\qquad\square$

The line bundles $\ell(1, 0)$ and $\ell(0, 1)$ depend on the initial values λ and u. By equation (4.16), we have

$$\ell(0, 1)_{\lambda, u} = n(\widetilde{E + eF}) + \left((\frac{1 - \lambda}{\lambda})n - u(2(1 - \lambda) + e\lambda) \right) \widetilde{F} + \frac{1}{2}\widetilde{K_S} - \mathcal{B}_0;$$

$$\ell(1, 0)_{\lambda, u} = \left((n - \frac{1}{2}e)(\frac{\lambda}{1 - \lambda}) + u(\lambda + (\frac{\lambda}{1 - \lambda})(e\lambda + 1 - \lambda)) \right) (\widetilde{E + eF})$$
$$+ \left(n - \frac{1}{2}e \right) n\widetilde{F} + \frac{1}{2}\widetilde{K_S} - \mathcal{B}_0.$$

By taking λ to 0^+ or 1^- respectively, we get two nef boundaries as (5.1):

$$\ell(0, 1)_{0^+, u} \xlongequal{\mathbb{R}_+} \widetilde{F}, \quad \ell(1, 0)_{1^-, u} \xlongequal{\mathbb{R}_+} (\widetilde{E + eF}).$$

Moreover, $\ell(0, 1)_{\lambda, u}$ is decreasing and $\ell(1, 0)_{\lambda, u}$ is increasing with respect to λ. The two type of loci are simultaneous contracted if and only if

$$\ell(0, 1)_{\lambda, u} = \ell(1, 0)_{\lambda, u}.$$

The solutions are given by

$$(5.2) \qquad\qquad \lambda = \frac{1}{2}, \text{ and } u = \frac{e}{e + 2}.$$

Theorem 5.2. *(a). [BC13, Theorem 1 (2.) (3.)] The nef cone* $\text{Nef}(\Sigma_e^{[n]})$ *($e \geq 0$, $n \geq 2$) is generated by the non-negative combinations of* $(\widetilde{E + eF})$, $\widetilde{F}$, *and* $(n-1)(\widetilde{E + eF}) + (n-1)\widetilde{F} - \frac{1}{2}\text{B}$.

(b). The nef cone $\text{Nef}(S_e^{[n]})$ *($e \geq 2$, $n \geq 2$) is generated by the non-negative combinations of* $(\widetilde{E + eF})$, $\widetilde{F}$, *and* $n(\widetilde{E + eF}) + (n-1)\widetilde{F} - \frac{1}{2}\text{B}$.

Proof. We only need to show that the Gieseker wall is not a higher rank wall in the case of (5.2). The two walls given by Lemma 5.1 coincide with center $C = -2n + \frac{e}{e+2}$. By the estimation formula [BC13, Section 5], the center C_k of a rank k wall ($k \geq 2$) is bounded by

$$C_k^2 \leq \left(u^2 + \frac{2n}{H^2}\right) \frac{(2k-1)^2}{(2k-1)^2 - 1} \leq \left(u^2 + \frac{2n}{H^2}\right) \frac{9}{8}.$$

Now $H^2 = \frac{e}{4u}$, and $u = \frac{e}{e+2}$. It is easy to check that

$$\left(u^2 + \frac{8nu}{e}\right)\frac{9}{8} < (-2n + u)^2.$$

So $C_k^2 < C^2$. Therefore higher rank walls are strictly inside the wall given by center C. By equation (4.16), the extremal nef line bundle corresponding to C is

$$n(\widetilde{E + eF}) + (n - \frac{e}{2})\widetilde{F} + \frac{1}{2}\widetilde{\text{K}_S} - \mathcal{B}_0.$$

The nef cone of $S^{[n]}$ is generated by the non-negative combinations of

$$(\widetilde{E + eF}), \quad \widetilde{F}, \quad n(\widetilde{E + eF}) + (n - \frac{e}{2})\widetilde{F} + \frac{1}{2}\widetilde{\text{K}_S} - \mathcal{B}_0.$$

Recall that $\mathcal{B}_0 = \frac{1}{2}\text{B}$ in Example 4.15. The proof is completed by using $\text{K}_S = -2(E + eF) + (e-2)F$ or $(e-2)F$ respectively. $\square$

The above computation suggests that the number u plays an important role in order to find the extremal nef line bundle, and in general $u \neq 0$, i.e. ω is not parallel to β. The nef cone of $\Sigma_e^{[n]}$ has been obtained by Bertram and Coskun [BC13]. The nef cone of $S_2^{[n]}$ ($n \geq 2$) has been obtained by J. Li and W.-P. Li [LL10]. Both of the results use the notion of k-very ample line bundles [BSG91].

Appendix A. Twisted Gieseker stability and the large volume limit

Definition A.1. [EG95, FQ95, MW97] Let $\omega, \alpha \in \text{NS}(S)_{\mathbb{Q}}$ with ω ample. For $E \in \text{Coh}(S)$, we denote the leading coefficient of $\chi(E \otimes \alpha^{-1} \otimes \omega^{\otimes m})$ with respect to m by a_d. A coherent sheaf E of dimension d is said to be α-twisted ω-Gieseker-(semi)stable if E is pure and for all $0 \neq F \subsetneq (\subseteq)E$,

$$(\text{A.1}) \qquad \frac{\chi(F \otimes \alpha^{-1} \otimes \omega^{\otimes m})}{a_d(F)} < (\leq) \frac{\chi(E \otimes \alpha^{-1} \otimes \omega^{\otimes m})}{a_d(E)} \quad \text{for } m \gg 0.$$

We also write them as (α, ω)-Gieseker (semi)stability. Denote $M_{(\alpha,\omega)}(\text{ch})$ (if exists) as the moduli space of (α, ω)-semistable sheaves E with $\text{ch}(E) = \text{ch}$.

Proposition-Definition A.2. ([LQ14]). Fix ch $= (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$. Fix a frame (H, γ, u) and consider $\sigma_{\omega,\beta}$ on the (s,t)-half-plane $\Pi_{(H,\gamma,u)}$ (Definition 2.3). Denote $s_0 := \frac{\mathrm{ch}_1.H}{\mathrm{ch}_0 H^2}$ if $\mathrm{ch}_0 \neq 0$. We always fix the relation

$$(\mathrm{A}.2) \qquad\qquad \alpha = \beta - \frac{1}{2}\mathrm{K}_S.$$

- **TC** If ch $= (0,0,n)$ with positive integer n, then there is *no wall*, and $t > 0$ is the *trivial chamber* in $\Pi_{(H,\gamma,u)}$. And

$$M_{\sigma_{\omega,\beta}}(\mathrm{ch}) = M_{(\alpha,\omega)}(\mathrm{ch}) = \mathrm{Sym}^n(S).$$

- **SC** If $\mathrm{ch}_0 = 0$, and $\mathrm{ch}_1.H > 0$, we define the chamber for $t \gg 0$ as the *Simpson chamber with respect to* (H, γ, u). Then

$$M_{\sigma_{\omega,\beta} \in \mathtt{SC}}(\mathrm{ch}) = M_{(\alpha,\omega)}(\mathrm{ch}).$$

  And the (α, ω)-Gieseker semistability is the Simpson semistability defined by the slope $\frac{\mathrm{ch}_2(E) - \mathrm{ch}_1(E).\beta}{\omega.\mathrm{ch}_1(E)}$.

- **GC** If $\mathrm{ch}_0 > 0$, we define the chamber for $t \gg 0$ and $s < s_0$ as the *Gieseker chamber with respect to* (H, γ, u). If ch satisfies condition (C), then

$$M_{\sigma_{\omega,\beta} \in \mathtt{GC}}(\mathrm{ch}) \cong M_{(\alpha,\omega)}(\mathrm{ch}).$$

- **UW** If $\mathrm{ch}_0 > 0$, we define the wall $t > 0$ and $s = s_0$, i.e. $\Im Z(\mathrm{ch}) = 0$ as the *Uhlenbeck wall with respect to* (H, γ, u).

- **DGC** If $\mathrm{ch}_0 < 0$, we define the chamber for $t \gg 0$ and $s > s_0$ as the *dual Gieseker chamber with respect to* (H, γ, u). If $-(\mathrm{ch})^*$ satisfies condition (C), then by Lemma 2.10,

$$M_{\sigma_{\omega,\beta} \in \mathtt{DGC}}(\mathrm{ch}) \cong M_{\sigma_{\omega,-\beta} \in \mathtt{GC}}(-(\mathrm{ch})^*).$$

- **DUW** If $\mathrm{ch}_0 < 0$, we define the wall $t > 0$ and $s = s_0$, i.e. $\Im Z(\mathrm{ch}) = 0$ as the *dual Uhlenbeck wall with respect to* (H, γ, u). If $-(\mathrm{ch})^*$ satisfies condition (C), then

$$M_{\sigma_{\omega,\beta} \in \mathtt{DUW}}(\mathrm{ch}) \cong U_\omega(-(\mathrm{ch})^*),$$

  where $U_\omega(-(\mathrm{ch})^*) = U_H(-(\mathrm{ch})^*)$ is the Uhlenbeck compactification of the moduli space $M_\omega^{\mathrm{lf}}(-(\mathrm{ch})^*)$ of locally free sheaves with invariant $-(\mathrm{ch})^*$.

Appendix B. Bayer-Macrì decomposition on K3 surfaces by using $\hat{\sigma}_{\omega,\beta}$

Let S be a smooth projective surface. By some physical hints (e.g. [Asp05, Section 6.2.3]), the central charge is often taken as (e.g. [Bri08, BM14a])

$$(\mathrm{B}.1) \qquad \hat{Z}_{\omega,\beta}(E) := -\int_S e^{-(\beta + \sqrt{-1}\omega)}.\mathrm{ch}(E).\sqrt{\mathrm{td}(S)}.$$

Similar as Lemma 2.2, one can check that

$$(\mathrm{B}.2) \qquad \hat{Z}_{\omega,\beta}(E) = \langle \mho_{\hat{Z}}, v(E) \rangle_S, \quad \text{where} \quad \mho_{\hat{Z}_{\omega,\beta}} := e^{\beta - \frac{1}{2}\mathrm{K}_S + \sqrt{-1}\omega}.$$

Write $\mho_{\hat{Z}_{\omega,\beta}}$ as $\mho_{\hat{Z}}$. Basic computation shows that

$$\langle \mho_Z, \mho_Z \rangle_S = \chi(\mathcal{O}_S) - \frac{1}{4}\mathrm{K}_S^2, \quad \langle \mho_{\hat{Z}}, \mho_{\hat{Z}} \rangle_S = -\frac{1}{8}\mathrm{K}_S^2.$$

Recall in [Bri09, BB16] that a numerical stability condition σ is called *reduced* if the corresponding $\pi(\sigma)$ satisfies $\langle \pi(\sigma), \pi(\sigma) \rangle_S = 0$.

In the following, we always *assume* that S is a smooth projective K3 surface, and *assume* that $\hat{Z}_{\omega,\beta}(F) \notin \mathbb{R}_{\leq 0}$ for all spherical sheaves $F \in \mathrm{Coh}(S)$. Then $\hat{\sigma}_{\omega,\beta} = (\hat{Z}_{\omega,\beta}, \mathcal{A}_{\omega,\beta})$ is a reduced numerical geometric Bridgeland stability condition [Bri08, Lemma 6.2]. Let $\mathbf{v} = v(\mathrm{ch}) \in H^*_{\mathrm{alg}}(S, \mathbb{Z})$ be a primitive class with $\langle \mathbf{v}, \mathbf{v} \rangle_S > 0$. Define $\hat{w}_{\omega,\beta} := \hat{w}_{\hat{\sigma}} := -\Im\left(\overline{\langle \mho_{\hat{Z}}, \mathbf{v} \rangle_S} \cdot \mho_{\hat{Z}} \right)$. Define $\ell_{\hat{\sigma},\mathcal{E}}$ the same as equation (3.1) by using $\hat{Z}$ instead. Then

$$\ell_{\hat{\sigma}_{\omega,\beta}} \xlongequal{\mathbb{R}_+} \theta_{\hat{\sigma},\mathcal{E}}(\hat{w}_{\omega,\beta}).$$

Fix a frame (H, γ, u). The potential walls $\hat{W}(\mathrm{ch}, \mathrm{ch}')$ in (s,t)-model are given by semicircles (or in (s,q)-model are given by semi-lines)

$$(\text{B.3}) \qquad (s - C)^2 + t^2 = C^2 + D + \frac{2}{H^2}, \quad (\text{or } q = Cs + \frac{1}{2}D + \frac{1}{H^2}),$$

where C and D are defined in Theorem 2.4. There is a global Bayer-Macrì map [BM14b, Theorem 1.2].

Theorem B.1. *(Bayer-Macrì decomposition on K3 surfaces.) Use notations and assumptions as above.*

- *If* $\mathrm{ch}_0 = 0$ *and* $\mathrm{ch}_1.H > 0$*, the Bayer-Macrì line bundle has a decomposition*

$$(\text{B.4}) \qquad \ell_{\hat{\sigma} \in \hat{W}(\mathrm{ch}, \mathrm{ch}')} \xlongequal{\mathbb{R}_+} \left(\frac{g}{2} D(\mathrm{ch}, \mathrm{ch}') + \frac{d}{2} u^2 \right) \mathcal{S} - \mathcal{T}_{(H, \gamma, u)}(\mathrm{ch}).$$

    The line bundle $\mathcal{S}$ induces the support morphism.

- *If* $\mathrm{ch}_0 > 0$*, the Bayer-Macrì line bundle has a decomposition*

$$(\text{B.5}) \qquad \ell_{\hat{\sigma} \in \hat{W}(\mathrm{ch}, \mathrm{ch}')} \xlongequal{\mathbb{R}_+} -C\widetilde{H} - u\widetilde{\gamma} - \mathcal{B}_0.$$

    The line bundle $\widetilde{\omega}$ (or $\widetilde{H}$) induces the Gieseker-Uhlenbeck morphism.

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